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Charged particles in a $(2+1)$-curved background

P Maraner

Center for Theoretical Physics, Laboratory for Nuclear Science and INFN, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, USA

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Abstract. The coupling to a (2+1)-background geometry of a quantized charged test particle in a strong magnetic field is analysed. Canonical operators adapting to the fast and slow freedoms produce a natural expansion in the inverse square root of the magnetic field strength. The fast freedom is solved to second order. At any given time, space is parametrized by a pair of conjugate operators and effectively behaves as the 'phase space' of the slow freedom. The slow Hamiltonian depends on the magnetic field norm, its covariant derivatives and the scalar curvature, and presents a peculiar coupling with the spin-connection.

1. Introduction

The dynamics of a charged particle in a given electromagnetic and gravitational background is an important problem having implications in several areas of theoretical and mathematical physics—from classical gravity to condensed matter and plasma physics to quantum field theory. As a quite interdisciplinary example, it represents the first step to take in addressing the study of a plasma around a compact astrophysical object or, more in general, in space and cosmological phenomena [1]. Exact solutions are found when metric and electromagnetic 2-form share common symmetries. Various special cases have been worked out, especially in two spatial dimensions, with particular emphasis on the underlying algebraic and analytical structures [2]. Beyond symmetry, in spite of the apparent simplicity, the general problem displays an extreme degree of complication. Classical motion is generally chaotic and one has to be content with approximate analysis. Even like this, however, the task to set down an appropriate perturbative expansion is not straightforward. The peculiar structure of the electromagnetic interaction makes ordinary Hamiltonian perturbation theory not directly applicable [3]. In the 1950s and 1960s, the urgency of the problem in classical applications, especially in connection with the investigation of the Earth's magnetosphere and in the design of mirror machines for the confinement of hot plasma, motivated Bogoliubov, Kruskal and others to formulate adiabatic perturbation theory directly in terms of the equations of motion. This led Northrop and Teller to the familiar 'guiding centre' picture of the effective dynamics of a charged particle in an inhomogeneous magnetic field in a flat spacetime [4]. Modern applications, ranging from geodesic motion around charged black holes in classical gravity to a two-dimensional system of non-relativistic electrons in quantum-Hall-like devices to plasma in astrophysics and cosmology to the investigation of the coupling with matter fields in toy models for quantum gravity, deal more in general with curved backgrounds and require the extension of the perturbative analysis developed in classical physics to the quantum-mechanical and field-theoretical context. To this task a whole

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canonical approach to the problem has to be developed. This is the aim of the present investigation.

In this paper we address the subject by discussing the effective motion of a charged particle in a $(2+1)$-curved background. This allows us to display the peculiar canonical structure of the system better, avoiding complications arising from extra dimensions. Moreover, the restriction is not just a mathematical artefact. The solution in two spatial dimensions is indeed a key ingredient in the discussion of the relativistic four-dimensional, as well as the non-relativistic three-dimensional, cases. From a rather different viewpoint the problem is also equivalent to the investigation of the effective dynamics of a test particle experiencing the 'geometric gravitational' force of Cangemi and Jackiw in a Wick-rotated two-dimensional spacetime [5].

Our analysis is based on the canonical structure of the system and is essentially the same for the classical and the quantum cases. For definiteness we consider the quantum case. The classical limit may be obtained straightforwardly. The topology of spacetime is supposed to be trivial—the direct product of a surface Σ diffeomorphic to the plane and time—so that all the local quantities automatically have a global definition; for example, Ricci rotation coefficients define a spin-connection. Under these hypotheses it is always possible to choose coordinates in such a way that the metric takes the form $g_{00}=1$, $g_{0\mu}=0$ and $g_{\mu\nu}$ are arbitrary functions of time and spatial coordinates, $\mu,\nu=1,2$. We also assume the electromagnetic field to be purely magnetic. Both relativistic and non-relativistic problems reduce then to the study of the Hamiltonian of a charged particle on the curved surface Σ.

This paper is organized as follows. In section 2 we discuss the canonical structure of the problem showing how the strong magnetic regime naturally produces an expansion in the inverse square root of the field strength. In section 3 an adapted set of canonical operators is introduced. This allows us to separate the fast freedom from the slow one, identifying the adiabatic invariant of the system. The coupling with background geometry is studied in section 4. Besides contributions depending on the scalar curvature and on covariant derivatives of the magnetic field norm, we find a peculiar coupling with the spin-connection. The theory is general as well as 'Lorenz' covariant. Our main result is the effective Hamiltonian (20). The example of a particle on a conical surface in an axisymmetric magnetic field decreasing as the inverse of the distance from the vertex is presented in section 5. The last section contains our conclusions. In the appendix the necessary technology for maximally simplifying the study of the adiabatic expansion is summarized.

2. Charged particle on a curved surface

We consider a charged scalar particle on a two-dimensional surface Σ in a *strong* magnetic background. The surface is parametrized by arbitrary coordinates x^μ, $\mu=1,2$, and its geometry is given by the metric tensor $g_{\mu\nu}$. The magnetic field is described by a closed antisymmetric 2-form $b_{\mu\nu}$. In both the non-relativistic and relativistic cases the discussion of the dynamical problem reduces to studying the Hamiltonian

$$\mathcal{H}=\tfrac{1}{2}g^{-1/2}\Pi_\mu g^{\mu\nu}g^{1/2}\Pi_\nu. \qquad (1)$$

The kinematical momenta $\Pi_\mu=-\mathrm{i}\partial_\mu-l_B^{-2}a_\mu$ have been introduced, $[\Pi_\mu,\Pi_\nu]=\mathrm{i}b_{\mu\nu}(\boldsymbol{x})$ and the physical dimension of the field is re-adsorbed in the scale factor l_B. The wavefunction of the system is normalized with respect to the measure $\sqrt{g}\,\mathrm{d}x^1\,\mathrm{d}x^2$. Our analysis is based on the smallness of the magnetic length l_B. Throughout our discussion we assume the background scalar curvature R as well as the derivatives of the magnetic field norm $b=\sqrt{b_{\mu\nu}b^{\mu\nu}/2}$ to satisfy the conditions $|R|\ll l_B^{-2}$, $|\triangle b/b|\ll l_B^{-2}$ and $|\nabla b/b|\ll l_B^{-1}$.

First, we focus on kinematics. In the absence of a magnetic interaction the essential operators appearing in the description of the system are the coordinates x^μ and the derivatives $-\mathrm{i}\partial_\mu$. These appear as a couple of conjugate variables, $[x^\mu,-\mathrm{i}\partial_\mu]=\mathrm{i}\delta^\mu_\nu$. Introducing the magnetic interaction replaces $-\mathrm{i}\partial_\mu$ by the non-commuting Π_μ. In other words, the magnetic background produces a twist of the canonical structure. This is made explicit by transforming to Darboux coordinate frames $\xi^\mu=\xi^\mu(x)$ in which the magnetic field strength takes the form

$$b_{\mu\nu}(\xi)=l_B^{-2}\varepsilon_{\mu\nu} \tag{2}$$

($\varepsilon_{\mu\nu}$ is the completely antisymmetric tensor in two dimensions). The Darboux theorem ensures the existence of a well defined atlas of such frames. In the new frames Π_1 and Π_2 appear as reciprocally conjugate while their commutators with the coordinates are still different from zero. On the other hand, $[\xi^1,\Pi_1]$ and $[\xi^2,\Pi_2]$ are order l_B^2 compared to $[\Pi_1,\Pi_2]$. This makes it clear that in the strong magnetic regime it is convenient to abandon the description in terms of ξ^μ and $-\mathrm{i}\partial_\mu$ introducing besides Π_1 and Π_2 a new pair of canonical variables. These turn out to be the guiding centre operators $\Xi^\mu=\xi^\mu+l_B^2\varepsilon^{\mu\nu}\Pi_\nu$ (in this paper we adopt the notation $\varepsilon^{\mu\nu}=\varepsilon_{\mu\nu}$). Rescaling for convenience Π_μ by $\Pi_\mu\to l_B\Pi_\mu$—and hence the magnetic field norm b and the Hamiltonian $\mathcal{H}$ by a factor l_B^2—the fundamental commutation relation may finally be re-cast in the form

$$[\Pi_1,\Pi_2]=\mathrm{i}\qquad [\Xi^2,\Xi^1]=\mathrm{i}l_B^2. \tag{3}$$

The presence of the small parameter l_B in the second relation displays the guiding centre operators as slow variables of the system. The physical interpretation of the new quantities emerges by considering dynamics in the semiclassical regime [6, 7]: the Π_μ take into account the rapid rotation of the particle while the Ξ^μ take into account the slow drift of the centre of the orbit, the guiding centre, on the surface.

Having outlined the peculiar canonical structure, we come back to the dynamical problem. This is in general of a certain complication†, the two freedoms of the system being coupled by the metric background $g_{\mu\nu}$ as well as by the magnetic field strength $b_{\mu\nu}$. Note that even starting from a simple geometrical context, for example a flat one, transforming to Darboux frames produces a quite complicated form of the interaction. Nevertheless, whenever the curvature radii of the surface and the variation length scale of the magnetic field may be considered larger than the magnetic length l_B, it turns out to be possible to perform an approximate analysis in quite general terms. We start, of course, from (1). As a first technical step we rescale the wavefunction and Hamiltonian by $\psi\to g^{1/4}\psi$ and $\mathcal{H}\to g^{1/4}\mathcal{H}g^{-1/4}$. This changes the integration measure from $\sqrt{g}\,\mathrm{d}x^1\,\mathrm{d}x^2$ to $\mathrm{d}x^1\,\mathrm{d}x^2$ making the Hamiltonian more symmetric and simplifying further manipulations. The second step is that of adapting variables. We transform, therefore, to Darboux coordinate frames according to the above kinematical discussion. The transformed metric tensor is denoted by $\gamma_{\mu\nu}$. Observe that in these preferential frames the metric determinant γ is related to the magnetic field norm by $\gamma=b^{-2}$. Moreover, all the functions of the coordinates have now to be evaluated in $\xi^\mu=\Xi^\mu-l_B\varepsilon^{\mu\nu}\Pi_\nu$ producing a natural expansion of the Hamiltonian

† An exact solution of the problem is only possible when the metric and magnetic field share a common symmetry. Typical examples are the Landau problem—motion on a plane in a uniform magnetic field—the motion on a sphere in a monopole field and the motion on the Poincaré half-plane in a hyperbolic magnetic field [2]. The exact (degenerate) ground state of the system may be obtained whenever the metric and magnetic 2-form define a Kähler structure on the surface Σ [8]. More general conditions require an approximate analysis.

in the small parameter l_B. Taking into account the rescaling of Π_μ, wavefunction and Hamiltonian, and expanding in l_B, (1) takes the form

$$l_B^2\mathcal{H} = \frac{1}{2}\gamma^{\mu\nu}\Pi_\mu\Pi_\nu - \frac{l_B}{2}(\partial_\kappa\gamma^{\mu\nu})\varepsilon^{\kappa\rho}\Pi_\mu\Pi_\rho\Pi_\nu + \frac{l_B^2}{4}(\partial_\kappa\partial_\lambda\gamma^{\mu\nu})\varepsilon^{\kappa\rho}\varepsilon^{\lambda\sigma}\Pi_\mu\Pi_\rho\Pi_\sigma\Pi_\nu - \frac{l_B^2}{4}\frac{\triangle b}{b} + \frac{l_B^2}{8}\frac{|\nabla b|^2}{b^2} + \mathcal{O}(l_B^3) \tag{4}$$

where the inverse metric $\gamma^{\mu\nu}$, the magnetic field norm b and all their derivatives are evaluated in the guiding centre operators Ξ^μ.

3. Spinning and drifting

We first focus on the zero order of expansion (4) by discussing the truncated Hamiltonian $\mathcal{H}^{(0)} = \frac{1}{2}\gamma^{\mu\nu}\Pi_\mu\Pi_\nu$. This is quadratic in Π_μ with coefficients depending on the slow variables Ξ^μ. It should therefore be possible to reduce the problem to a harmonic oscillator up to higher orders in l_B. To this task we consider the decomposition of $\gamma^{\mu\nu}$ in terms of the zwei-beinen ${e_i}^\mu$; $\gamma^{\mu\nu} = {e_i}^\mu {e_i}^\nu$. We then introduce the 'normalized zwei-beinen' ${n_i}^\mu = b^{-1/2}{e_i}^\mu$ in such a way that

$$\mathcal{H}^{(0)} = \tfrac{1}{2}[{n_i}^\mu(\Xi)\Pi_\mu]b(\Xi)[{n_i}^\nu(\Xi)\Pi_\nu] - \tfrac{1}{4}\mathrm{i}{n_i}^\mu(\Xi)b(\Xi){n_i}^\nu(\Xi)\varepsilon_{\mu\nu}. \tag{5}$$

It is clear that the new $\bar\Pi_\mu$ recasting $\mathcal{H}^{(0)}$ in a harmonic oscillator Hamiltonian should have the form $\bar\Pi_i = {n_i}^\mu(\Xi)\Pi_\mu + \mathcal{O}(l_B^2)$. To obtain a genuine set of canonical variables—not a perturbative one—we produce the rotation of Π_μ in the ${n_i}^\mu$ directions by means of the unitary transformation

$$U = \exp\{-\tfrac{1}{4}\mathrm{i}\varepsilon^{\mu i}{[\ln n]_i}^\nu\{\Pi_\mu, \Pi_\nu\}\}. \tag{6}$$

The new canonical operators are defined by $\bar\Pi_i = \delta_i^\mu U\Pi_\mu U^\dagger$ and $\bar\Xi^\mu = U\Xi^\mu U^\dagger$. An explicit expression as a power series in l_B^2 may now be obtained to any order. As a preparation for the next section we write the new variables to order l_B^2. Introducing the non-covariant rotation coefficients ${\rho_i}^j{}_{,k} = {n_k}^\mu(\partial_\mu {n_i}^\nu){n_\nu}^j$ we have

$$\bar\Pi_i = {n_i}^\mu\Pi_\mu - \tfrac{1}{8}l_B^2\varepsilon^{mn}{\rho_i}^k{}_{,m}{\rho_j}^l{}_{,n}\varepsilon^{jh}{n_k}^\kappa{n_h}^\mu{n_l}^\lambda(\Pi_\kappa\Pi_\mu\Pi_\lambda + \Pi_\lambda\Pi_\mu\Pi_\kappa) + \mathcal{O}(l_B^4) \tag{7}$$

$$\bar\Xi^\mu = \Xi^\mu - \tfrac{1}{4}l_B^2\varepsilon^{mn}{n_m}^\mu{\rho_j}^l{}_{,n}\varepsilon^{jk}{n_k}^\kappa{n_l}^\lambda(\Pi_\kappa\Pi_\lambda + \Pi_\kappa\Pi_\lambda) + \mathcal{O}(l_B^4) \tag{8}$$

where all the functions on the right-hand side are evaluated in Ξ. In order to rewrite (5) in terms of the new operators, these relations have to be inverted. The task is straightforward yielding

$${n_i}^\mu(\Xi)\Pi_\mu = \bar\Pi_i + \tfrac{1}{8}l_B^2\varepsilon^{mn}{\rho_i}^k{}_{,m}{\rho_j}^l{}_{,n}\varepsilon^{jh}(\bar\Pi_k\bar\Pi_h\bar\Pi_l + \bar\Pi_l\bar\Pi_h\bar\Pi_k) + \mathcal{O}(l_B^4) \tag{9}$$

and

$$\Xi^\mu = \bar\Xi^\mu + \tfrac{1}{4}l_B^2\varepsilon^{mn}{n_m}^\mu{\rho_j}^l{}_{,n}\varepsilon^{jk}(\bar\Pi_k\bar\Pi_l + \bar\Pi_l\bar\Pi_k) + \mathcal{O}(l_B^4) \tag{10}$$

where in both equations the functions on the right-hand side are now evaluated in $\bar\Xi$. The substitution of (9) and (10) into (5) produces the zero-order Hamiltonian as a power series in l_B^2. Introducing the harmonic oscillator $J = \frac{1}{2}(\bar\Pi_1^2 + \bar\Pi_2^2)$, we obtain

$$\mathcal{H}^{(0)} = b(\bar\Xi)J + l_B^2\mathcal{H}^{(0,2)}(\bar\Xi, \bar\Pi) + \mathcal{O}(l_B^4) \tag{11}$$

where $\mathcal{H}^{(0,2)}$ is a quite complicated expression, quartic in $\bar\Pi_\mu$ and depending on $\bar\Xi^\mu$ through b and ${\rho_i}^j{}_{,k}$, which may be evaluated by direct substitution.

The adiabatic behaviour of the system in the strong magnetic regime may now be read in the first term of expansion (11). The fast and slow freedoms decouple up to higher order in l_B. The fast freedom is frozen in one of the harmonic oscillator eigenstates of the adiabatic invariant J. While 'spinning', the particle drifts on the surface Σ. The drifting is Hamiltonian: the configuration space Σ appears now as the phase space of the slow freedom; the magnetic field norm $b(\bar{\Xi}^1, \bar{\Xi}^2)$ expressed by the pair of conjugate variables $\bar{\Xi}^1$ and $\bar{\Xi}^2$ is the Hamiltonian operator governing the slow motion (see [9]).

The situation is substantially analogous to the motion on a plane [7, 10], the metric appearing only in the evaluation of the magnetic field norm. The crucial difference is that in a non-trivial geometrical background a constant value of b does not produce, in general, the slow variables as exact constants of motion.

4. Coupling to background geometry

We now study the higher-order corrections to the effective motion of the charged particle. To this task we proceed by the so-called averaging method (see the appendix), that is by performing a series of near-identity unitary transformations separating, order by order in l_B, the fast freedom from the slow freedom. First, a little preparation is necessary.

We re-express all the quantities appearing in expansion (4) in terms of the new canonical variables $\bar{\Pi}_i$ and $\bar{\Xi}^\mu$. This produces the replacements of all the curved space indices $\mu, \nu, \ldots$ by the flat space indices $i, k, \ldots$. Every 'general covariant' index μ is replaced by a 'Galilei covariant' index i according to the usual rules $v_i = e_i{}^\mu v_\mu$, $v^i = e^i{}_\mu v^\mu$ etc.

As a second step it is useful to work out a few basic geometrical identities holding in every Darboux frame. These will be precious in bringing the adiabatic expansion into an explicit covariant form. By taking the derivative of the relation between the metric determinant and magnetic field norm we obtain $(\partial_\rho \gamma^{\mu\kappa})\gamma^{\nu\lambda}\varepsilon_{\kappa\lambda} - (\partial_\rho \gamma^{\nu\kappa})\gamma^{\mu\lambda}\varepsilon_{\kappa\lambda} = 2b(\partial_\rho b)\varepsilon^{\mu\nu}$. Contracting with $\varepsilon_{\mu\nu}$ and rewriting in terms of flat space indices yields

$$\Gamma^j_{ij} = -b^{-1}(\partial_i b) \tag{12}$$

(which is the usual relation $\Gamma^\nu_{\mu\nu} = \partial_\mu \ln g^{1/2}$ evaluated in a Darboux frame). By multiplying the relation by itself, contracting and rewriting in terms of flat space indices we also obtain

$$\Gamma^k_{ij}\Gamma^k_{ij} - \Gamma^k_{ij}\Gamma^j_{ik} - \Gamma^j_{ij}\Gamma^k_{ik} + 2\Gamma^j_{ii}\Gamma^k_{jk} - \Gamma^k_{ii}\Gamma^k_{jj} = 0. \tag{13}$$

No other general relations hold among the various contractions of the Christoffel symbols.

We proceed now by evaluating the contributions produced by the zero-, first- and second-order terms of (4). Everywhere in what follows equation (13) is used to eliminate $\Gamma^k_{ii}\Gamma^k_{jj}$ in favour of the other four possible contractions of the Christoffel symbols.

4.1. Second-order contribution from $\mathcal{H}^{(0)}$

We start the averaging procedure considering the second-order contribution produced by $\mathcal{H}^{(0,2)}$. To this task it is necessary to re-express the non-geometrical quantities $\rho_i{}^j{}_{,k}$ in terms of the spin-connection $\omega_{ij,k} = (\nabla_{e_k} e_i) \cdot e_j$ and the Christoffel symbols Γ^k_{ij}. A quick computation yields

$$b^{1/2}\rho_i{}^j{}_{,k} = \omega_i{}^j{}_{,k} + \tfrac{1}{2}\delta^i_j\Gamma^l_{kl} - \Gamma^j_{ik}. \tag{14}$$

We recall that the spin-connection is completely antisymmetric in the indices i and j. In two dimensions it may, therefore, be rewritten in terms of a $U(1)$ gauge potential as $\omega_{ij,k} = \omega_k \varepsilon_{ij}$. A point-dependent rotation by an angle $\chi(\xi)$ of the zwei-beinen $e_i{}^\mu$ produces the gauge transformation $\omega_k \to \omega_k + \partial_k \chi$. By replacing $\rho_i{}^j{}_{,k}$ in $\mathcal{H}^{(0,2)}$ according to (14), the second-order contribution to the perturbative expansion is readily evaluated by the formula (A1):

$$\mathcal{H}^{(0)} \longrightarrow (-\varepsilon^{ij}\Gamma^k_{ik}\omega_j + \tfrac{1}{4}\Gamma^k_{ij}\Gamma^j_{ik} + \tfrac{1}{2}\Gamma^j_{ij}\Gamma^k_{ik} - \tfrac{3}{4}\Gamma^j_{ii}\Gamma^k_{jk})J^2 + \tfrac{3}{16}\Gamma^k_{ij}\Gamma^j_{ik} - \tfrac{3}{16}\Gamma^j_{ii}\Gamma^k_{jk}. \tag{15}$$

Quite surprisingly a term explicitly depending on ω_k survives.

4.2. *Second-order contribution from $\mathcal{H}^{(1)}$*

The first-order term of expansion (4) is cubic in the kinematical momenta, $\mathcal{H}^{(1)} = -\frac{1}{2}b^{1/2}(\partial_l\gamma^{ij})\varepsilon^{lk}\bar{\Pi}_i\bar{\Pi}_k\bar{\Pi}_j$. As shown in the appendix, this contributes to the adiabatic expansion an l_B^2 order term that may be directly evaluated by means of (A2). The only necessary preparation is that of re-expressing $\partial_k\gamma^{ij}$ in terms of Christoffel symbols. This is done by rewriting $\partial_\kappa\gamma_{\mu\nu}$ in terms of $\partial_\kappa\gamma^{\mu\nu}$ in the definition $\Gamma^\rho_{\mu\nu}$ and by symmetrizing. The contraction with the zwei-beinen produces

$$\partial_k\gamma^{ij} = -(\Gamma^i_{jk} + \Gamma^j_{ik}). \tag{16}$$

By substitution into (A2) we obtain

$$\mathcal{H}^{(1)} \longrightarrow (-\tfrac{3}{4}\Gamma^k_{ij}\Gamma^k_{ij} - \tfrac{3}{4}\Gamma^k_{ij}\Gamma^j_{ik} - \tfrac{3}{4}\Gamma^j_{ij}\Gamma^k_{ik} + \tfrac{3}{2}\Gamma^j_{ii}\Gamma^k_{jk})J^2 - \tfrac{3}{16}\Gamma^k_{ij}\Gamma^k_{ij} - \tfrac{7}{16}\Gamma^k_{ij}\Gamma^j_{ik} + \tfrac{1}{16}\Gamma^j_{ij}\Gamma^k_{ik} + \tfrac{3}{8}\Gamma^j_{ii}\Gamma^k_{jk}. \tag{17}$$

4.3. *Second-order contribution from $\mathcal{H}^{(2)}$*

A similar computation has to be performed for the second-order term of the perturbative expansion, $\mathcal{H}^{(2)} = \frac{1}{4}(\partial_m\partial_n\gamma^{ij})\varepsilon^{mk}\varepsilon^{nl}\bar{\Pi}_i\bar{\Pi}_k\bar{\Pi}_l\bar{\Pi}_j$. This time it is necessary to re-express the second-order derivatives of the inverse metric in terms of the Christoffel symbols and their derivatives. This is simply obtained by taking the derivative of (16)

$$\partial_m\partial_n\gamma^{ij} = -\partial_m\Gamma^i_{nj} - \partial_m\Gamma^j_{ni} + \Gamma^h_{im}\Gamma^j_{hn} + \Gamma^h_{jm}\Gamma^i_{hn} + \Gamma^i_{mh}\Gamma^j_{nh} + \Gamma^j_{mh}\Gamma^i_{nh}. \tag{18}$$

Recalling the definition of the scalar curvature $R = \partial_i\Gamma^i_{jj} - \partial_i\Gamma^j_{ij} + \Gamma^j_{ii}\Gamma^k_{jk} - \Gamma^k_{ij}\Gamma^j_{ik}$, formula (A1) yields the second-order contribution produced by $\mathcal{H}^{(2)}$

$$\mathcal{H}^{(2)} \longrightarrow (\tfrac{1}{4}R - \tfrac{1}{4}\partial_i\Gamma^j_{ij} + \tfrac{3}{4}\Gamma^k_{ij}\Gamma^k_{ij} + \tfrac{1}{2}\Gamma^k_{ij}\Gamma^j_{ik} - \tfrac{1}{4}\Gamma^j_{ij}\Gamma^k_{ik} - \tfrac{1}{2}\Gamma^j_{ii}\Gamma^k_{jk})J^2 - \tfrac{1}{16}R - \tfrac{5}{16}\partial_i\Gamma^j_{ij} + \tfrac{3}{16}\Gamma^k_{ij}\Gamma^k_{ij} + \tfrac{1}{4}\Gamma^k_{ij}\Gamma^j_{ik} + \tfrac{1}{16}\Gamma^j_{ij}\Gamma^k_{ik} + \tfrac{1}{8}\Gamma^j_{ii}\Gamma^k_{jk}. \tag{19}$$

The two terms still containing derivatives of the Christoffel symbols may be expressed in terms of derivatives of the magnetic field norm b and contractions of Γ^k_{ij} simply by taking the derivative of equation (12), $\partial_i\Gamma^j_{ij} = -b^{-1}\triangle b + b^{-2}|\nabla b|^2 + \Gamma^j_{ii}\Gamma^k_{jk}$.

4.4. *Effective dynamics*

The effective Hamiltonian describing the motion of a charged particle to second order in the adiabatic parameter l_B is finally obtained by adding to $b(\bar{\Xi})J$ the contributions (15), (17) and (19) as well as the term $-\nabla b/4b + |\nabla b|^2/8b^2$. As one must expect, all the contractions of Γ^k_{ij} except $\Gamma^j_{ij}\Gamma^k_{ik}$ cancel. This may be rewritten in terms of ∇b by means of (12). We obtain

$$\mathcal{H} = \frac{bJ}{l_B^2} + \left(\frac{1}{4}R + \frac{\nabla b}{b}\times\omega + \frac{1}{4}\frac{\triangle b}{b} - \frac{3}{4}\frac{|\nabla b|^2}{b^2}\right)J^2 - \frac{1}{16}R + \frac{1}{16}\frac{\triangle b}{b} - \frac{1}{16}\frac{|\nabla b|^2}{b^2} + \mathcal{O}(l_B). \tag{20}$$

All the functions are expressed by the pair of conjugate operators $\bar{\Xi}^1$ and $\bar{\Xi}^2$. As before this expression has to be interpreted as the effective Hamiltonian describing the motion of the slow freedom while the particle is frozen in one of the J eigenstates. The second term is the correction which survives in the classical limit while the remaining ones are of a pure quantal nature.

Even if our computation has been carried out in a Darboux coordinate frame, equation (20) is explicitly covariant so that we are free to transform back to the original—arbitrary—coordinates x^μ. The price to pay is that of dealing with non-canonical operators, the Hamiltonian being evaluated in $X^\mu = x^\mu(\bar{\Xi})$. These 'guiding centre variables' in fact satisfy the non-canonical commutation relations $[X^2, X^1] = \mathrm{i} l_B^2 b^{-1}(X)$ (see [6, 7]).

The effective dynamics is sensitive to the background scalar curvature. This coupling is particularly relevant when the magnetic 2-form is proportional—in arbitrary coordinates—to the volume 2-form, $b_{\mu\nu} = l_B^{-2}\sqrt{g}\varepsilon_{\mu\nu}$. $g_{\mu\nu}$ and $b_{\mu\nu}$ then define a Kähler structure on Σ. The particle interacts only with the surface. The magnetic force becomes the 'geometric gravitational' force of Cangemi and Jackiw [5]. In the strong magnetic regime the effective Hamiltonian driving the slow motion is proportional to the scalar curvature. In the semiclassical regime test particles drift along the line of constant curvature of the surface Σ.

The effective dynamics is coupled to the background spin-connection as well. The coupling is not explicitly gauge invariant. A gauge transformation $\omega_k \to \omega_k + \partial_k\chi$ adds the term $l_B^2 b^{-1}\varepsilon^{ij}(\partial_i b)(\partial_j\chi)J^2$ to the Hamiltonian. Gauge invariance may nevertheless be restored by the unitary transformation $U = \mathrm{e}^{\mathrm{i}b^{-1}J\chi}$. The second-order term $-\mathrm{i}b^{-1}[b(\bar{\Xi}), \chi(\bar{\Xi})]J^2$ produced in this way produces (20) in its original form.

Last but not least, it is worth mentioning that expansion (20) yields the correct flat limit [6, 7] supplying a full canonical derivation of it.

5. Around a conical singularity

A typical situation of interest in (2+1) gravity is the motion around a conical singularity [11]. As an example we consider, therefore, a charged particle on a conical surface subjected to an axisymmetric magnetic field decreasing as the inverse of the distance from the vertex. The problem is explicitly solvable, allowing a check upon our strong magnetic field expansion.

The cone is parametrized by the distance ρ from the vertex, in the range $0 \leqslant \rho \leqslant +\infty$, and the angle ϕ, in the range $0 \leqslant \phi \leqslant 2\pi$; the points $\phi = 0$ and $\phi = 2\pi$ are identified. In these coordinates the metric and magnetic 2-forms take the form

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha^2\rho^2 \end{pmatrix} \qquad b_{\mu\nu} = \frac{1}{l_B^2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{21}$$

where α is the conical angle; setting $\alpha = 1$ brings the cone in the Euclidean plane. Although the curvature of $\gamma_{\mu\nu}$ vanishes, the geometry of the space is non-trivial. The spin-connection of the surface reads $\omega_\mu = (0, \alpha)$ and cannot be gauged away for non-integer values of α.

We first consider the exact solution. In order to have a deeper insight into the problem we focus on the classical motion, the discussion of the quantum problem proceeding essentially along the same lines. Choosing the vector potential as $a_\mu = (0, l_B^{-2}\rho)$ the Hamiltonian of the system is written as

$$\mathcal{H} = \frac{1}{2}p_\rho^2 + \frac{1}{2\alpha^2\rho^2}\left(p_\phi - \frac{\rho}{l_B^2}\right)^2. \tag{22}$$

Given axial symmetry, the momentum p_ϕ is conserved, $\{\mathcal{H}, p_\phi\} = 0$, and it can be replaced by its constant value L. The radial motion of the system takes place in the effective Keplerian potential

$$V_{\text{eff}}(\rho) = \frac{L^2}{2\alpha^2\rho^2} - \frac{L}{\alpha^2 l_B^2 \rho} + \frac{1}{\alpha^2 l_B^4} \tag{23}$$

where $L/\alpha^2 l_B^2$ appears as an attractive Newton constant, L/α as the angular momentum and the whole spectrum is shifted by the energy $1/\alpha^2 l_B^4$. The presence of the magnetic field produces bound states in the system. There is no need to go through the well known solution of this problem, we focus instead on the qualitative behaviour of the system in the strong magnetic regime. For small values of l_B the minimum $\bar{\rho} = L l_B^2$ of the effective potential becomes extremely deep and narrow. V_{eff} is very well approximated by a harmonic oscillator centred in $\bar{\rho}$ and with frequency $\omega = 1/\alpha L l_B^4$. While rotating around the axis of the cone at a distance $\bar{\rho}$, the particle performs very rapid oscillations. The result is that of a very thin and dense spiral wrapping around an orbit of constant radius. Neglecting the rapid oscillation, the effective angular velocity may be evaluated by eliminating L in favour of $\bar{\rho}$ in the relation $L = p_\phi = \alpha^2\rho^2\dot{\phi}$. This yields

$$\dot{\phi} \approx \frac{1}{\alpha^2 l_B^2 \bar{\rho}}. \tag{24}$$

The angular velocity distribution gives information on the conical angle α.

We now come to the strong magnetic expansion (20). Observe that the coordinates ρ and ϕ are already of Darboux type. The rapid oscillations of the particle have obviously to be identified with the freedom $\Pi_\rho - \Pi_\phi$, while the drift on the cone can be identified with the motion of the guiding centre variables $R = \rho + l_B \Pi_\phi$ and $\Phi = \phi - l_B \Pi_\rho$. The coordinates ρ, ϕ and the pair of conjugate variables R, Φ parametrize two phase-space surfaces very close to each other and may be confused when orders higher than l_B are neglected. The Hamiltonian driving the effective motion is immediately obtained from (20) by evaluating the gradient and Laplacian of the magnetic field norm $b(\rho) = 1/(\alpha l_B^2 \rho)$; the Galilei covariant components of the spin-connection are given by $\omega_i = (0, 1/\rho)$;

$$\mathcal{H}_{\text{eff}} = \frac{1}{\alpha l_B^2 R} J - \frac{1}{2R^2} J^2 + \cdots. \tag{25}$$

The angle Φ does not appear in the Hamiltonian so that R is a second constant of motion besides J. The particle moves around the axis at a constant value of the radius $\bar{R}$. The angle Φ evolves linearly in time according to the Hamilton equations $\dot{\Phi} = J/\alpha\bar{R} + \mathcal{O}(l_B^2)$. The adiabatic invariants J and R are directly related by the conservation of energy. Recalling that the classical system is in the adiabatic regime for small values of the total energy, we re-obtain an angular velocity distribution with the behaviour (24).

6. Conclusions

The purpose of this paper was to show how it is possible to set down a systematic canonical perturbative analysis for the motion of charged particles in a curved background geometry. This bridges the gap between classical canonical theory and non-canonical averaging methods traditionally employed in classical analysis. Most importantly, the method allows a direct discussion of the quantum case extending to this realm the whole classical 'guiding centre' picture. The aim is essentially achieved by means of Darboux transformations, standard averaging methods and elementary differential geometry. For the sake of simplicity, we have restricted our attention to $(2+1)$ dimensions. Aside from its importance in the discussion of the whole $(3+1)$-dimensional problem, the $(2+1)$-dimensional system is already of a certain applicative importance in itself. An immediate application concerns the investigation of the non-minimal coupling of Cangemi and Jackiw in a Wick-rotated two-dimensional gravity. More in general, Hamiltonian (20) gives us immediate information on how wavefunctions and eigenvalues of an electron in a quantum-Hall-like device are modified when a small inhomogeneity of the magnetic field or of the thin film geometry are introduced. The electron behaves like a one-degree-of-freedom system having the thin film—the spatial surface Σ—as 'phase space'. The fast freedom is still frozen in a harmonic oscillator eigenstate and the discussion of section 3 indicates how the harmonic oscillator eigenfunctions have to be constructed. The peculiar way the slow freedom couples to the 'phase space' scalar curvature and spin-connection is particularly intriguing and deserves further investigation. Another important issue concerns the convergence of the perturbative expansion, which is expected, in general, to be, an asymptotic series. We conclude by pointing out that considering more spatial dimensions produces other interesting phenomena—for example, the coupling of the effective dynamics of the new freedoms with geometry-induced gauge structures—that can be described essentially by the same formalism. The inclusion of spin is also quite immediate. The restriction to $(2+1)$ dimensions allowed us to single out the effective coupling with the background geometry without mixing it with phenomena of a different nature. The effective motion in $(3+1)$ dimensions and the inclusion of spin will be considered in future publications.

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Appendix. Averaging around a harmonic oscillator

The introduction of a suitable set of variables reduces the study of the effective motion of a charged particle to the discussion of a Hamiltonian of the form

$$\mathcal{H} = \alpha J + \epsilon\alpha^{ijk}\Pi_i\Pi_k\Pi_j + \epsilon^2\alpha^{ijkl}\Pi_i\Pi_k\Pi_l\Pi_j + \cdots .$$

As in the main text, $J = \frac{1}{2}(\Pi_1^2 + \Pi_2^2)$, and Π_1, Π_2 are a pair of conjugate variables, $[\Pi_1, \Pi_2] = \mathrm{i}$. ϵ is a small parameter. The coefficients appearing in the expansion are allowed to depend on slow variables. The self-adjointness of $\mathcal{H}$ requires $\alpha^{ijk} = \alpha^{jik}$ and $\alpha^{ijkl} = \alpha^{jilk}$. For a charged particle in a strong magnetic field the coefficients α, α^{ijk}, α^{ijkl}, ... are quite complicated expressions involving the spin-connection, the metric tensor and their derivatives evaluated in the slow guiding centre variables Ξ^1 and Ξ^2, $[\Xi^2, \Xi^1] = \mathrm{i}\epsilon^2$. Very useful formulae will be worked out in this appendix in order to maximally simplify the manipulation of these expressions.

When ϵ is set equal to zero, the dynamics is described by $h^{(0)} \equiv \alpha J$. The system behaves as a harmonic oscillator with frequency depending on the non-dynamical parameters Ξ^i. A non-zero value of ϵ turns the perturbation on, making, at the same time, the guiding centre operators into a couple of conjugate dynamical variables. In order to extract the effective dynamical content of the theory to the various orders in the perturbative parameter ϵ, we will subject the system to a series of near-identity unitary transformations. These are chosen in such a way that the various terms of the perturbative expansion depend on Π_1 and Π_2 only though J and its powers. This makes J into an adiabatic invariant—a quantity conserved up to higher order of some power of ϵ—and allows us to identify the Hamiltonian driving the effective motion of the slow variables in correspondence with every value taken by J. The technique is based essentially on the identity $\mathrm{e}^{\mathrm{i}a}\mathcal{H}\,\mathrm{e}^{-\mathrm{i}a} = \mathcal{H} + \mathrm{i}[a, \mathcal{H}] - \frac{1}{2}[a, [a, \mathcal{H}]] + \cdots$, where $a = 1 + \epsilon a^{(1)} + \epsilon^2 a^{(2)} + \cdots$ is the generator of a near-identity unitary transformation. The self-adjoint operators $a^{(1)}$, $a^{(2)}$, etc have to be chosen order by order in such a way that the desired conditions are matched.

We start by the order ϵ of the expansion: $h^{(1)} \equiv \alpha^{ijk}\Pi_i\Pi_k\Pi_j$. Note that since $\Pi_i\Pi_k\Pi_j + \Pi_j\Pi_k\Pi_i$ is completely symmetric in the indices i, j and k, only the completely symmetric part of α^{ijk} matters. We can therefore assume the complete symmetry of α^{ijk}. It is then easy to verify that the choice

$$a^{(1)} = -\tfrac{1}{3}\alpha^{-1}(\alpha^{ijl} + 2\delta^{ij}\alpha^{hhl})\varepsilon_{lk}\Pi_i\Pi_k\Pi_j$$

produces the counterterm $\mathrm{i}[a^{(1)}, h^{(0)}] = -h^{(1)}$. The first-order term of the transformed expansion vanishes identically. The operation is nevertheless not painless. The transformation in fact contributes the second-order term $h^{(1,2)} = \frac{1}{2}\mathrm{i}[a^{(1)}, h^{(1)}]$. This can be evaluated in

$$\begin{aligned} h^{(1,2)} = \frac{3}{2}\left(\frac{\alpha^{ihh}\alpha^{jkl}}{\alpha} + \frac{\alpha^{jhh}\alpha^{ikl}}{\alpha} - \frac{\alpha^{ijh}\alpha^{klh}}{\alpha} - 2\frac{\delta^{ij}\alpha^{hhg}\alpha^{klg}}{\alpha}\right)\Pi_i\Pi_k\Pi_l\Pi_j \\ -\frac{\alpha^{ijk}\alpha^{ijk}}{\alpha} + 3\frac{\alpha^{iik}\alpha^{jjk}}{\alpha}. \end{aligned}$$

The problem is reduced to the discussion of the second-order term.

We focus therefore on $h^{(2)} \equiv \alpha^{ijkl}\Pi_i\Pi_k\Pi_l\Pi_j$. The symmetrization in the various pairs of indices can still be performed producing contributions of the form $\alpha^{ijkl}\varepsilon_{ij}\varepsilon_{kl}$ etc, not depending on Π_1 and Π_2. Nevertheless, a quite useful expression can already be obtained by assuming only the symmetrization of the first and second pairs of indices, which is the case we have to deal with. A brief computation then shows that the right choice to ensure that the second order of the perturbative expansion depends only on powers of J is

$$a^{(2)} = -\tfrac{1}{8}\alpha^{-1}[\alpha^{ijkh} + \delta^{ij}(\alpha^{kghg} + \tfrac{1}{2}\alpha^{khgg})]\varepsilon^{hl}\Pi_i\{\Pi_k, \Pi_l\}\Pi_j.$$

The second-order term $\mathrm{i}[a^{(2)}, h^{(0)}]$ produced by this transformation combines with $h^{(2)}$ in such a way to give the final contribution to the perturbative expansion

$$\alpha^{ijkl}\Pi_i\Pi_k\Pi_l\Pi_j \longrightarrow (\alpha^{ijij} + \tfrac{1}{2}\alpha^{iijj})J^2 - \tfrac{1}{4}\alpha^{ijij} + \tfrac{5}{8}\alpha^{iijj}. \tag{A1}$$

No matter how complicated is $h^{(2)}$, formula (A1) allows us to immediately write down the contribution to the effective dynamics by evaluating a few contractions of the coefficients α^{ijkl}. The first application of (A1) is the second-order contribution produced by $h^{(1)}$ through $h^{(1,2)}$. A brief computation yields the quite compact formula

$$\alpha^{ijk}\Pi_i\Pi_k\Pi_j \longrightarrow -\left(\frac{3}{2}\frac{\alpha^{ijk}\alpha^{ijk}}{\alpha} + \frac{9}{4}\frac{\alpha^{iik}\alpha^{jjk}}{\alpha}\right)J^2 - \frac{5}{8}\frac{\alpha^{ijk}\alpha^{ijk}}{\alpha} + \frac{3}{16}\frac{\alpha^{iik}\alpha^{jjk}}{\alpha}. \tag{A2}$$

Again, the contribution to the effective dynamics produced by $h^{(1)}$ may be obtained through (A2) by evaluating a few contractions on the square of the coefficients α^{ijk}.

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